Tensors, ranks, and varieties.

Enrico Carlini

Department of Mathematical Sciences
Politecnico di Torino, Italy

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Tensor rank

Let $V_1, \ldots, V_m$ be $\mathbb{C}$-vector spaces of dimension $\dim V_i = n_i + 1$.

A tensor $T \in V = V_1 \otimes \ldots \otimes V_m$ is

$$T = \sum \alpha_{i_1,\ldots,i_m} v_{i_1} \otimes \ldots \otimes v_{i_m}$$

where the coefficients $\alpha_{i_1,\ldots,i_m} \in \mathbb{C}$ and the vectors $v_{i_j} \in V_j$.

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where the coefficients $\alpha_{i_1,\ldots,i_m} \in \mathbb{C}$ and the vectors $v_{i_j} \in V_j$.

There are some distinguished elements in V that we commonly use to represent all the other elements

Elementary tensors

A tensor

$$v_1 \otimes \ldots \otimes v_m \in V$$

with $v_i \in V_i$ is called *elementary tensor*.

Tensor rank

Note that using elementary tensors we can construct a basis for V and thus for any $T \in V$ we can write

$$T = \sum_{i=1}^{r} E_i$$

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The *tensor rank* of T is

$$\mathrm{rk}(T) = \min\{r : T = \sum_{i=1}^{r} E_i, E_i \text{ elementary}\}.$$

Tensor rank

Example $V = V_1 \otimes V_2$

In this case $T \in V$ can be written as

$$T = \sum_{i,j} \alpha_{ij} v_i \otimes v_j.$$

Fixing bases in V_1 and V_2, T corresponds to the $\dim V_1 \times \dim V_2$ matrix

$$A_T = (\alpha_{ij}).$$

Elementary tensors correspond to matrices of **rank one**, thus

$$\mathrm{rk}(T) = \mathrm{rk}(A_T).$$

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corresponds to a tensor

$\mathbf{M}_{\langle \mathbf{n},\mathbf{m},\mathbf{p} \rangle}$

$$\mathbf{M}_{\langle \mathbf{n},\mathbf{m},\mathbf{p} \rangle} \in \mathbb{C}^{n,m*} \otimes \in \mathbb{C}^{m,p*} \otimes \mathbb{C}^{n,p}$$

is the *matrix multiplication tensor*. If $n = m = p$, that is for square matrices, we just write $\mathbf{M}_{\langle \mathbf{n} \rangle}$.

Knowing $\mathrm{rk}(\mathbf{M}_{\langle \mathbf{n},\mathbf{m},\mathbf{p} \rangle})$ relates to the computational complexity of matrix multiplication.

Tensor rank

It is not difficult to find an upper bound for $\mathrm{rk}(\mathbf{M}_{\langle \mathbf{n},\mathbf{m},\mathbf{p}\rangle})$.

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Given matrices

$$A = (a_{ij}) \in \mathbb{C}^{n,m}, B = (b_{jl}) \in \mathbb{C}^{m,p}, C = (c_{il}) \in \mathbb{C}^{n,p}$$

and choosing dual bases $\{\alpha_{ij}\}$ and $\{\beta_{jl}\}$ we get that

$$\mathbf{M}_{\langle \mathbf{n},\mathbf{m},\mathbf{p}\rangle} = \sum_{ijl} \alpha_{ij} \otimes \beta_{jl} \otimes c_{il}$$

and thus the conclusion follows.

For example $\mathrm{rk}(\mathbf{M}_{\langle \mathbf{n}\rangle}) \leq n^3$.

Tensor rank

Strassen's result and $\mathbf{M}_{\langle 2 \rangle}$

The usual matrix multiplication in the case 2×2 is

$$\mathbf{M}_{\langle 2 \rangle} \in \mathbb{C}^{2,2} \otimes \in \mathbb{C}^{2,2} \otimes \mathbb{C}^{2,2}$$

where

$$\mathbf{M}_{\langle 2 \rangle} = \sum_{i=1}^{8} E_i$$

for eight elementary tensors and thus

$$\mathrm{rk}(\mathbf{M}_{\langle 2 \rangle}) \leq 8.$$

But in the '60s Strassen wanted to prove that equality holds and...

Tensor rank

Strassen's result and $\mathbf{M}_{\langle 2\rangle}$

Strassen showed that

$$\mathrm{rk}(\mathbf{M}_{\langle 2\rangle}) \leq 7,$$

and we now know that equality holds. That is

$$\mathbf{M}_{\langle 2\rangle} = \sum_{i=1}^{7} F_i$$

for **seven**, and **no fewer**, elementary tensors F_i. Thus one can multiply $n \times n$ matrix with complexity

$$O(n^{\log_2 7}).$$

Tensor rank

$\mathbf{M}_{\langle 3 \rangle}$

Clearly

$$\mathrm{rk}(\mathbf{M}_{\langle 3 \rangle}) \leq 27,$$

and we know that

$$19 \leq \mathrm{rk}(\mathbf{M}_{\langle 3 \rangle}) \leq 23,$$

but we do not know the actual value yet!

X-rank

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We want to find a uniform setting to deal with different notions of ranks (e.g. tensor rank, symmetric rank). First we note that our rank definition are invariant up to scalar multiplication, thus it is natural to work over the *projective space*.

Projective space

Given a $N+1$ dimensional vector space V, we define

$$\mathbb{P}(V) = \mathbb{P}^N \setminus 0 = V/\mathbb{C}^*$$

and $[v] \in \mathbb{P}(V)$ is the equivalence class $\{\lambda v : \lambda \in \mathbb{C} \setminus \{0\}\}$.

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V(I)

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Note that to each algebraic variety $X \subseteq \mathbb{P}^N$ corresponds a *radical ideal*

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- The image of an algebraic projective variety via a polynomial map is a projective variety
- Algebraic varieties are the closed subset of the Zariski topology

X-rank

Given an algebraic variety $X \subset \mathbb{P}^N$ and a point $p \in \mathbb{P}^N$, we define

X-rank

The X-rank of p with respect to X is

$$X\mathrm{-rk}(p) = \min\{r : p \in \langle p_1, \ldots, p_r \rangle, p_i \in X\}$$

where

$$\langle p_1, \ldots, p_r \rangle = \mathbb{P}(\{\lambda_1 v_1 + \ldots + \lambda_r v_r : \lambda_i \in \mathbb{C}\})$$

is the linear span of the points $p_i = [v_i]$'s.

Clearly, $X\mathrm{-rk}(p) = 1$ if and only if $p \in X$.

X-rank

Segre varieties

Given vector spaces $V_1, \dots, V_t$, we consider the map

$$s : \mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_t) \longrightarrow \mathbb{P}(V_1 \otimes \dots \otimes V_t)$$

$$[v_1], \dots, [v_t] \mapsto [v_1 \otimes \dots \otimes v_t]$$

this is called the *Segre map* and its image X is called the *Segre product* of the varieties $\mathbb{P}(V_i)$.

X-rank

Segre varieties

Since the Segre variety $X = s(\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_t))$ parameterizes elementary tensors in $V_1 \otimes \ldots \otimes V_t$, it is clear that

$$X\mathrm{-rk}([T]) = \min\{r : [T] \in \langle [E_1], \ldots, [E_r] \rangle\}$$

and thus the X-rank with respect to the Segre variety is just the (tensor) rank.